

AUTOMORPHIC COHOMOLOGY OF MUMFORD-TATE DOMAINS

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ABSTRACT. The article discusses some general motivations from Hodge theory toward automorphic representations. We explain the semistability of Hodge bundles and the parabolicity of these bundles. We end up with interesting ideas between Mumford-Tate groups of Hodge structures and non abelian group cohomology.

INTRODUCTION

Consider a variation of Hodge structure

$$\Phi : S \rightarrow \Gamma_{\mathbb{Z}} \backslash D$$

where S is a smooth base manifold and Γ is a discrete group. Φ may also be regarded as the period map associated to a family of polarized Hodge structure on a vector space V defined over \mathbb{Q} . D is called a period domain and it is simply known that it is a hermitian symmetric complex manifold. There are naturally defined Hodge bundles F^p of the Hodge structure on V , and also the endomorphism bundle $\mathfrak{g} = \text{End}(V)$ on D . The corresponding local systems are $\mathcal{V} := \Gamma \backslash (D \times \mathcal{V})$ and $\mathcal{G} := \Gamma \backslash (D \times \mathfrak{g})$, respectively. Lets, denote the Hermitian metric by h to be any of the induced metric from that of the upper half plane H . One way to explain the complex structure on D is to embed it in its compact dual \check{D} , which is the set of all Hodge filtrations on V with the same Hodge numbers satisfying the first Riemann-Hodge bilinear relation. \check{G} is a homogeneous complex manifold. There are $G_{\mathbb{C}}$ -homogeneous vector bundles $F^p \rightarrow \check{D}$, called Hodge bundles whose fiber at a given point F^{\bullet} is F^p . Over $D \subset \check{D}$ we have $V^{p,q} = F^p / F^{p+1}$, which are homogeneous vector bundles for the action of $G_{\mathbb{R}}$. They are Hermitian vector bundles with $G_{\mathbb{R}}$ -invariant Hermitian metric given in each fiber by the polarization form. The space of functions on D can be identified with the $\Gamma_{\mathbb{Z}}$ -automorphic functions on D . A basic example of this is a variation $V = V^{1,0} \oplus V^{0,1}$ obtained from the middle cohomology of an elliptic fibration of curves. In this case $D = H$ is the upper half plane, $\Gamma_{\mathbb{Z}} = Sl_2(\mathbb{Z})$ and the functions

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on the Hodge domain $Sl_2(\mathbb{Z}) \backslash H$ are the usual modular forms, that are holomorphic functions on H satisfying;

$$(1) \quad f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^n f(\tau)$$

Around a cusp one sets $q = e^{2\pi i\tau}$ and expands in Laurent series $f(q) = \sum_n a_n q^n$. The finiteness condition at the cusps are then $a_n = 0$ for $n < 0$. If we consider the Deligne extension $V_e^{n,0}$ the the modular forms are sections of $V_e^{n,0} \rightarrow \Gamma \backslash H$ that extend to $H_e^{n,0} \rightarrow \overline{\Gamma \backslash H}$. A modular form is a cusp form if $a_0 = 0$, i.e it vanishes at the cusps. This condition is equivalent to $\int_{\Gamma \backslash H} \|\psi\| d\mu < \infty$, [2].

Let f, g be two cusp forms of weight n on H the upper half plane. By putting

$$(2) \quad \langle f, g \rangle = \int_{\Gamma \backslash H} f(z) \overline{g(z)} y^{n-2} dx dy, \quad x = Re(z), y = Im(z)$$

one obtains a Hermitian scalar product on the space of modular forms which is positive and non-degenerate, invariant by $\Gamma = PSl_2(\mathbb{Z})$. One can check that $\langle T(k)f, g \rangle = \langle f, T(k)g \rangle$, where $T(k)$ is the k -th Hecke operator defined by

$$(3) \quad T(k).f = k^{n-1} \sum_{\substack{ad=k \\ a \geq 0, 0 \leq b < d}} d^{-n} f\left(\frac{az + b}{d}\right)$$

A vector valued $\mathcal{A}^{p,q}$ -automorphic form $f \in \mathcal{A}^{p,q}(D, \mathfrak{g})$ satisfies the following condition;

$$(4) \quad f(\gamma.z) \gamma'(z)^p \overline{\gamma'(z)^q} = Ad(\rho) f(z), \quad \gamma \in \Gamma$$

where $\rho : \pi_1 \rightarrow Gl(V)$ is the representation of the monodromy on the parameter base space. $\mathcal{A}^{p,q}(D, \mathfrak{g})$ is vector space of smooth C^∞ -forms on D with values in \mathfrak{g} . The Peterson inner product between two modular forms on the upper half plane can be naturally stated for the vector valued automorphic forms. The hyperbolic metric descended from H defines the Hodge $*$ -operator on \mathcal{V} and \mathcal{G} . Then the Laplace operator and the harmonic forms can be defined in the same fashion as in Hodge theory. The hermitian metrics $h_{\mathcal{V}}$ and $h_{\mathcal{G}}$ define the Hodge star operators on the corresponding space of smooth (p, q) -forms. The Laplace operator is defined by $\Delta = \bar{\partial}^* \bar{\partial}$ where $\bar{\partial}^* := - * \bar{\partial} *$. It is a self adjoint operator on the space of L^2 -sections of the corresponding bundles. The identification $\mathcal{V} := \Gamma \backslash (H \times \mathcal{V})$ and $\mathcal{G} := \Gamma \backslash (H \times \mathfrak{g})$ identify these Laplace operators with the Laplace operator on H acting on the space of Γ -automorphic functions. The Hodge inner product is

$$(5) \quad \langle f, g \rangle = \int_H f(z) \wedge *g(z) dx dy = \int_{\Gamma \backslash H} \text{tr}(f(z) \wedge g(z)^*) y^{p+q-2} dx dy$$

where $f^* = \bar{f}^t$. In the case of usual automorphic functions the right hand side is Peterson inner product (2). This may be regarded as a hermitian form induced from the hyperbolic metric on the upper half plane, [5].

The aforementioned identification of \mathcal{V} and \mathcal{G} identifies this inner product with the one induced by the Hodge inner product, defined by the Hodge star operator. If we add this fact from the mixed Hodge metric theorem that the polarization form on the Hodge structures are unique, it follows that the polarization form and the one induced from Peterson inner product must be the same up to a constant factor.

1. MODULI OF HODGE STRUCTURES AND AUTOMORPHIC COHOMOLOGY

Let $h^{p,q}$ be the Hodge numbers of a Hodge structure

$$\phi : \mathbb{U}(\mathbb{R}) \rightarrow \text{Aut}(V_{\mathbb{R}})$$

of weight n and period domain D with compact dual \check{D} . The group $G_{\mathbb{R}} = \text{Aut}(V_{\mathbb{R}}, Q)$ is a real simple Lie group that acts transitively on D . The isotropy group H of a reference polarized Hodge structure (V, Q, ϕ) would be a compact subgroup of $G_{\mathbb{R}}$ that contains a compact maximal torus T . One has

$$(6) \quad D = \{\phi : S^1 \rightarrow G_{\mathbb{R}} ; \phi = g^{-1} \phi_0 g\}$$

It follows that $H = Z_{\phi_0}(G_{\mathbb{R}})$, the centralizer of $\phi_0(S^1)$. An easy exercise in linear algebra shows

$$(7) \quad H \cong \begin{cases} U(h^{2m+1}) \times \dots \times U(h^{m+1,m}) & n = 2m + 1 \\ U(h^{2m}) \times \dots \times U(h^{m+1,m-1}) \times \mathcal{O}(h^{m,m}) & n = 2m \end{cases}$$

The group $G_{\mathbb{C}}$ is a complex simple Lie group that acts transitively on \check{D} . The subgroup P that stabilizes a F_0^\bullet is a parabolic subgroup with $H = G_{\mathbb{R}} \cap P$. The case $n = 1$ is classical and one knows that $D = H_g$ the Siegel generalized upper half space $= \{Z \in M_{g \times g} : Z = {}^t Z, \text{Im}(Z) > 0\}$.

The Lie algebra \mathfrak{g} of the simple Lie group $G_{\mathbb{C}}$ is a \mathbb{Q} -linear subspace of $\text{End}(V)$, and the form Q induces on \mathfrak{g} a non-degenerate symmetric bilinear form

$$(8) \quad B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$$

which upto scale is just the Cartan-Killing form $\text{tr}(\text{ad}(x)\text{ad}(y))$. For each point $\phi \in D$

$$(9) \quad \text{Ad}\phi : \mathbb{U}(\mathbb{R}) \rightarrow \text{Aut}(\mathfrak{g}_{\mathbb{R}}, B)$$

is a Hodge structure of weight 0 on \mathfrak{g} . This Hodge structure is polarized by B .

Associated to each nilpotent transformation $N \in \mathfrak{g}$ one defines a limit mixed Hodge structure. The local system $\mathfrak{g} \rightarrow \Delta^*$ is then equipped with the monodromy $T = e^{\text{ad}N}$ and Hodge filtration defined with respect to the multi-valued basis of \mathfrak{g} by $e^{\log(t)\frac{N}{2\pi i}}F^\bullet$. It gives a limit MHS $(\mathfrak{g}, F^\bullet, W(N)_\bullet)$. The polarizing form gives perfect pairings

$$(10) \quad B_k : Gr_k^{W(N)} \mathfrak{g} \times Gr_{-k}^{W(N)} \mathfrak{g} \rightarrow \mathbb{Q}, \quad B_k(u, v) = B(v, N^k v)$$

defined via the hard Lefschetz isomorphism $N^k : Gr_{-k}^{W(N)} \mathfrak{g} \cong Gr_k^{W(N)} \mathfrak{g}$, [2], [3].

Proposition 1.1. *The Hodge polarization of the Hodge bundles \mathcal{V} and \mathcal{G} are given by the Peterson Inner product as discussed in Sec (1). The Peterson inner product induces the Cartan-Killing form $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$ on the fibers of \mathcal{G} , and also the graded forms $\langle \cdot, \cdot \rangle_k : PGr_k^{W(N)} \mathfrak{g} \times PGr_{-k}^{W(N)} \mathfrak{g} \rightarrow \mathbb{Q}$, $\langle \cdot, \cdot \rangle_k := \langle \mathfrak{l}^k u, v \rangle = \langle v, \mathfrak{l}^k v \rangle$ when the fibration is involved with the monodromy $T = e^{\text{ad}(N)}$, where \mathfrak{l} is induced from $\text{ad}(N)$. Moreover the action of the Hecke operators induce self adjoint linear transformations on \mathfrak{g} which can be identified as $T(k) = \mathfrak{l}_k$. The vector space \mathcal{S} of complex automorphic forms (representations) with coefficients in a Hodge bundle on a period (Mumford-Tate) domain has a mixed Hodge structure which is polarized by the above form. (Riemann-Hodge bilinear relations for automorphic forms) Let the primitive subspaces of \mathcal{S} be defined by*

$$P_l : \ker(N^{l+1}) : Gr_l^W \mathcal{S} \rightarrow Gr_{-l-2}^W \mathcal{S}$$

Have a pure Hodge structure

$$P_l = \bigoplus_{p+q=k+l} S^{p,q}$$

of weight $k+l$ (for some $k > 0$) polarized by the bilinear form $\langle \cdot, \cdot \rangle_l : P_l \times P_l \rightarrow \mathbb{C}$ induced by Peterson inner product and satisfies

- $\langle \psi, \eta \rangle_l = 0$, $\psi \in S^{p,q}, \eta \in S^{r,s}$ unless $r = p, s = q$.
- $* \times \langle \psi, C_l \cdot \psi \rangle_l > 0$ for any $\psi \neq 0$ in $S^{p,q}$, where C is the corresponding Weil operator and $*$ is a complex constant.

Proof. The proof follows from the discussion in section (1) and (2) and the uniqueness of the polarization form for mixed Hodge structures as well of the operators \mathfrak{l}_k known as mixed Hodge metric. \square

We try to apply the above situation to the cohomology of period domains, where the cohomology classes are automorphic forms. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C} = \text{Lie}(G_{\mathbb{R}} \otimes \mathbb{C})$ be a complex semi-simple Lie algebra, $\mathfrak{h} = \mathfrak{t} \otimes \mathbb{C}$ a Cartan subalgebra, $\mathfrak{t} = \text{Lie}(T)$, and $K_{\mathbb{C}}$ a complex Lie group corresponding to the unique maximal compact subgroup $K \subset G_{\mathbb{R}}$. We denote $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ to be the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. We assume the action of $K_{\mathbb{C}}$ will be locally finite, and its differential agrees with the corresponding subspace of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$. One may match these data with the case $D = G_{\mathbb{R}}/H$ is a general Mumford-Tate domain siting in the diagram

$$(11) \quad \begin{array}{ccc} G_{\mathbb{R}}/T & \longrightarrow & G_{\mathbb{C}}/B \\ \downarrow & & \downarrow \\ D = G_{\mathbb{R}}/H & \longrightarrow & G_{\mathbb{C}}/P = \check{D} \end{array}$$

with T a maximal torus, B a Borel subgroup, and horizontal arrows to be inclusions, [2].

Definition 1.2. A Harish-Chandra (HC)-module M is a $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$ -module that is finite as $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ -module with an admissible $K_{\mathbb{C}}$ -action.

We propose to show when the automorphic cohomology $H^q(D, L_{\mu})$ of a period domain gives a HC-module. Let $\mathcal{H} = \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$. The Weyl group W of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ acts on \mathcal{H} and gives an isomorphism $Z(\mathfrak{g}_{\mathbb{C}}) \cong \mathcal{H}^W$, where the upper-index means the elements fixed by W . For each $\mu \in \mathfrak{h}_{\mathbb{C}}^*$ the homomorphism $\chi_{\mu} : Z(\mathfrak{g}_{\mathbb{C}}) \rightarrow \mathbb{C}$, $z \mapsto z(\mu)$ is called an infinitesimal character. A result of Harish-Chandra says that any character of $Z(\mathfrak{g}_{\mathbb{C}})$ is an infinitesimal character, and $\chi_{\mu} = \chi_{\mu'}$ iff $\mu = w(\mu')$ for some $w \in W$. Lets fix a set of positive roots Φ^+ of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Note that we have

$$(12) \quad \mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{n}^+ \oplus \mathfrak{n}^-, \quad \mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}^{\alpha}$$

Any root $\mu \in \Phi^+$ defines an integrable almost complex structure on $G_{\mathbb{R}}/T$ as well as a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^-$ and a line bundle $L_{\mu} = G_{\mathbb{C}} \times_B \mathbb{C}$. The sections of a vector bundle $\mathcal{E} = G_{\mathbb{R}} \times_T E \rightarrow D$ already defined by the representation $r : T \rightarrow \text{Aut}(E)$ are E -valued functions $f : G_{\mathbb{R}} \rightarrow E$ such that the right action of T is given by $f(g.t) = r(t^{-1})f(g)$ where \cdot . Now take

$$(13) \quad \mathcal{E}_{\mu} = \Lambda^q T^{0,1*} D \otimes L_{\mu}$$

Identifying $T_s^{0,1} = \mathfrak{n}$ obtain $A^{0,q}(D, L_\mu) = (C^\infty(G_\mathbb{R}) \otimes \Lambda^q \mathfrak{n}^* \otimes L_\mu)^T$. We abbreviate this by $(C^\infty(G_\mathbb{R}) \otimes \Lambda^q \mathfrak{n}^*)_{-\mu}$. Summarizing the identifications of the complexes of $G_\mathbb{R}$ -modules we get

$$(14) \quad A^{0,q}((D, L_\mu), \bar{\partial}) = (C^\infty(G_\mathbb{R}) \otimes \Lambda^q \mathfrak{n}^*, \delta), \quad H^q(D, L_\mu) = H^q(\mathfrak{n}, C^\infty(G_\mathbb{R}))_{-\mu}$$

See [2] for details.

Theorem 1.3. [GS] *Let $\rho = \sum_{\alpha \in \Phi^+} \alpha/2$. If $\mu + \rho$ is singular, i.e. $\exists \alpha \in \Phi$, $(\mu + \rho, \alpha) = 0$, then*

$$H^k(D, \mathcal{E}) = 0, \quad \forall k$$

If $\mu + \rho$ is nonsingular, then let ω be the element of the Weyl group that carries $\mu + \rho$ into the highest Weyl chamber. Let $l = \#\{\alpha \in \Delta_+ \mid \omega(\alpha) \text{ is negative}\}$. Then

$$H^k(D, \mathcal{E}) = \begin{cases} 0 & k \neq l \\ \mathcal{E}_{\omega(\mu+\rho)-\rho}, & k = l \end{cases}$$

If $X = \Gamma \backslash D$, and \mathcal{F} be the induced automorphic vector bundle on X from \mathcal{E} , then the corresponding automorphic cohomology satisfies

$$H^k(Y, \mathcal{F}) = \begin{cases} 0, & k \neq \alpha(\pi) \\ \mathbb{C}^r, & k = \alpha(\pi) \end{cases}$$

where $r = c(D)v(X) \dim(E_{\omega(\mu+\rho)-\rho})$, the constant $c(D)$ only depends to D , $v(D)$ is volume of the fundamental domain of Γ , and

$$\alpha(\pi) = \#\{\alpha \in \Delta_+ - \Phi \mid (\lambda, \alpha > 0\} + \#\{\alpha \in \Delta_+ - \Phi \mid (\lambda, \alpha < 0\}$$

It is well-known that the cohomology ring $H^*(D, \mathcal{E})$ is generated by the chern classes of Hodge bundles on D . Such a chern class can be written as a constant multiple of

$$\Theta_D = B \wedge^t \bar{B} - C \wedge^t \bar{C}$$

$$B = \sum_{\rho, \alpha \in \Delta_{\mathfrak{v}} - \Phi, (\lambda, \alpha) > 0} \nu_\rho \otimes e_\alpha \omega_\rho^* \otimes \omega^\alpha$$

$$C = \sum_{\sigma, \alpha \in \Delta_{\mathfrak{v}} - \Phi, (\lambda, \alpha) < 0} \nu_\sigma \otimes e_\alpha \omega_\sigma^* \otimes \omega^\alpha$$

where B, C, B', C' are matrices of $(1, 0)$ -forms. e_α , $\alpha \in \Delta$ are root vectors of G , ω^α are dual roots, λ is the highest weight. ω_ρ is a basis for \mathfrak{g} which is an extension of a

basis on \mathfrak{k} , Φ is defined via the root decomposition of the subalgebra $\mathfrak{k} = \mathfrak{h} \oplus_{\beta \in \Phi} \mathfrak{g}_\beta$. Finally ω_ρ^* is the dual basis and $e_\alpha \omega_\rho^*$ is the contragredient representation, [GS].

2. HODGE BUNDLES VS STABLE PARABOLIC BUNDLES

Let X be a compact connected Riemann surface of genus g and $\emptyset \neq S \subset X$ a divisor. A parabolic bundle E with parabolic divisor S consists of the data of a filtration (quasi-parabolic condition) $E_s = E_{s,1} \supset \dots \supset E_{s,l_s} \supset 0$, $\forall s \in S$, and rational numbers $0 \leq \alpha(s) < \dots < \alpha_{l_s} < 1$ called parabolic weights. Fix a positive integer r . The parabolic degree of E is defined by

$$(15) \quad \text{Par-deg}(E) = \deg(E) + \sum_{s \in S} \sum_i \alpha_{i(s)}(s) \dim(E_{s,i}/E_{s,i+1})$$

Then every sub-bundle of E is parabolic in a natural way. We call a parabolic bundle semi-stable if the factor (par-deg/rank) is non-increasing when passing to sub-bundles. In case this number is decreasing for every proper sub-bundles we call the bundle stable. A Hermitian structure on a parabolic bundle E is a Hermitian structure on $E|_{X \setminus S}$ with the extra condition that around each $s \in S$ for any section σ of E defined around s , if $\sigma(s)$ is non-zero in $E_{s,i}$, then

$$(16) \quad \|\sigma\| = f(z)|z|^{\alpha_i(s)}$$

with positive real valued f . The definition of a parabolic bundles can be generalized for the higher dimensional base. In this case the set S is a divisor. One may apply this definition to the Hodge filtration on period domains, see the Proposition 2.2 of this section.

Theorem 2.1. (*Mehta-Seshadri*) *A parabolic bundle E of rank k and parabolic degree 0 is stable if and only if it is isomorphic to a bundle E^ρ , where $\rho : \Gamma \rightarrow U(k)$ is an irreducible unitary representation of the group Γ admissible with respect to the weights and multiplicities of the parabolic structure of E . Moreover, parabolic bundles E^{ρ_1} and E^{ρ_2} are isomorphic if and only if the representations ρ_1 and ρ_2 are equivalent.*

A unitary representation $\rho : \Gamma \rightarrow U(k)$ is called admissible with respect to a given set of weights and multiplicities at S if for each i , we have $\rho(S_i) = U_i D_i U_i^{-1}$, where S_i is a generator for the local monodromy at $P_i \in S$, with unitary $U_i \in U(k)$ and $D_i = \exp(2\pi i \cdot \text{diag}[\alpha_1^i, \dots, \alpha_{r_i}^i])$, where each $\alpha_l^i = \alpha_l(P_i)$ is repeated $k_l^i = k_l(P_i)$ times.

Proposition 2.2. [5] *The bundles $\mathcal{V}_D := \mathcal{V} \otimes \mathcal{O}_D$, $\mathcal{G}_D := \mathcal{G} \otimes \mathcal{O}_D$ as well as the Hodge bundles \mathcal{F}^p are parabolic bundles over D .*

Proof. (sketch) Admissible matrices are parametrized by the flag varieties $F_i = U(k)/U(k_1) \times U(k_2) \times \dots \times U(k_{r_i})$. The group Γ acts on the trivial bundle $H \times \mathbb{C}^k$ by the rule $(z, \gamma) \mapsto (\gamma z, \rho(\gamma)v)$. Take the sheaf of its bounded (Γ, ρ) -sections around the cusps. The direct image of this sheaf under $H \rightarrow X$ is a locally free sheaf of rank k . The parabolic structure at the image of cusps is defined by the matrices $\rho(S_i)$. This gives a parabolic vector bundle on the Riemann-Surface X . Loosely speaking this bundle is the extension of the bundle $E^\rho = \Gamma \backslash (H \times \mathbb{C}^k) \rightarrow \Gamma \backslash H$. This proves the parabolicity of \mathcal{V} . The proof for \mathcal{G} is similar. The result for the Hodge bundles follows from the trivial fact that the holomorphic subbundles of a parabolic bundle are also parabolic. \square

The standard Hermitian metric in \mathbb{C}^k defines a Γ -invariant metric on the trivial vector bundle $H \times \mathbb{C}^k \rightarrow H$. It extends as a (pseudo)-metric to the bundle $E = E^\rho$. Explicitly, we choose $\sigma_i \in Sl_2(\mathbb{R})$ such that $\sigma_i(\infty) = x_i$ and $\sigma_i^{-1} S_i \sigma_i = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$, and make the change of coordinates $\zeta = e^{2\pi i \sigma_i^{-1} z}$ at $P_i \in \Gamma \backslash H$. Then the metric h is given by a diagonal matrix $(\zeta^{2\alpha_1^i}, \dots, \zeta^{2\alpha_{r_i}^i})$, [5].

Another question we are interested in is to study the variation of the (almost) complex structure on the homogeneous space D . For this we make the following general set up. Let G be a reductive Lie group defined over \mathbb{R} and $\mathfrak{g} := Lie(G) = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_{\alpha}$ its root decomposition, with $\alpha \in R$ and T a maximal torus in $G_{\mathbb{C}}$. Also consider $V \subset G$ a closed Lie subgroup with

$$(17) \quad \mathfrak{v} = Lie(V_{\mathbb{C}}) = \mathfrak{h} \oplus \bigoplus_{\beta \in \Delta} \mathfrak{g}_{\beta}$$

Any almost complex structure on G/V gives

$$J : \mathfrak{g}_{\mathbb{R}}/\mathfrak{v}_{\mathbb{R}} \rightarrow \mathfrak{g}_{\mathbb{R}}/\mathfrak{v}_{\mathbb{R}}, \quad J^2 = -id$$

$$J(\theta(v).X) = \theta(v).J(X), \quad v \in V, \quad X \in \mathfrak{g}_{\mathbb{R}}/\mathfrak{v}_{\mathbb{R}},$$

where θ is the isotropy representation on V . The restriction of the isotropy representation θ to T is the adjoint representation giving (17). It follows that J leaves each isotropycal \mathfrak{g}_{β} invariant, and reduces to multiplication by $\pm i$ on \mathfrak{g}_{β} , $\beta \in R - \Delta$. Set

$$(18) \quad C := \{\alpha \in R - \Delta \mid J|_{\mathfrak{g}_{\alpha}} = i \cdot id\}$$

A choice of G -invariant almost complex structure on G/V gives rise to such a C such that $\alpha \in R - \Delta$ implies only one of α or $-\alpha$ is in C . It is possible to choose a system of positive roots R_+ such that $C \subset R_+$. In this case R_+ is called the system of positive roots adopted to invariant almost complex structure. This criteria generalizes to

arbitrary homogeneous vector bundles on G/V associated to a representation $\rho : V \rightarrow \mathrm{Gl}(H)$ on a vector space H . One can apply the above criteria to the system of Hodge bundles on D or \tilde{D} and obtain a sequence of subsets C_p of the R adopted to the whole filtration, $C_p \subset C_{p-1}$. The theorem states that this procedure is also reversible. That is any choice of this adopted series provides complex structure on each F^p .

Proposition 2.3. [4] *Let $\rho : V \rightarrow \mathrm{Gl}(H)$ be an n -dimensional complex representation and $E := G \times_V E$ the associated vector bundle on G/V . Then there exists a 1-1 correspondence between G -invariant almost complex structures on E and*

$$\mathrm{Hom}_V^J(\bigoplus_{\alpha \in C} \mathfrak{g}_\alpha, \mathfrak{g}') := \{f \mid f \circ J = -J \circ f\}$$

where $\mathfrak{g}' = \mathfrak{gl}_n(\mathbb{C})$, with a choice of C satisfying (18). Moreover such almost complex structure, will be always a complex (holomorphic) structure.

The proof of this theorem follows from a careful diagram chasing in the natural split exact sequence of V -modules

$$(19) \quad 0 \rightarrow \mathfrak{g}' \rightarrow \mathfrak{p} = (\mathfrak{g} \times \mathfrak{g}')/\mathfrak{h}_{\mathbb{R}} \rightarrow (\mathfrak{g}/\mathfrak{v})_{\mathbb{R}} \rightarrow 0$$

The G -invariant holomorphic structures then correspond to all possible extensions of $\mathrm{ad}\rho : \mathfrak{v} \rightarrow \mathfrak{gl}(H)$ to $\mathrm{ad}\rho : \mathfrak{v} \oplus_{\alpha \in C} \mathfrak{g}_{-\alpha} \rightarrow \mathfrak{gl}(H)$. In the case of Hodge bundles the complex structures of the system of Hodge bundles correspond the number successive extensions when you add the roots in $C_{p-1} \setminus C_p$ at each step of the Hodge filtration.

Theorem 2.4. [4] *Let G be a reductive connected Lie group defined over \mathbb{R} , and V a closed subgroup such that G/V is an irreducible Hermitian symmetric space. Let ρ be any irreducible representation of V . Then the associated homogeneous vector bundle is stable.*

Theorem 2.4 states that for any holomorphic subbundle L of F^p we have

$$\dim(L) \cdot c_1(F^p) - \dim(F^p) \cdot c_1(L) > 0$$

Applying this theorem to the situation of period domains we get,

Corollary 2.5. *The bundles $\mathcal{V}_D, \mathcal{G}_D$ as well as the Hodge bundles \mathcal{F}^p are stable vector bundles over D .*

3. NON-ABELIAN COHOMOLOGY AND MT-GROUPS OF HODGE STRUCTURES

(1) Let G be a group and A (not necessarily abelian) another group on which G acts on the left. Write A multiplicatively. $H^0(G, A)$ is by definition the group A^G of elements of A fixed by G . A 1-cocycle would be a map $s \mapsto a_s$ from $G \rightarrow A$ such that $a_{st} = a_s \cdot s(a_t)$. Two cocycles a_s and b_s are equivalent if there exists $a \in A$ such that $b_s = a^{-1} \cdot a_s \cdot s(a)$ for all $s \in G$. This defines an equivalence relation and the quotient set is denoted by $H^1(G, A)$. It is a pointed set with a distinguished element of the unit cocycle $a_s = 1$. Here $s(-)$ means the action of $s \in G$, and a_s is the value of the cocycle a at $s \in G$. These two definitions agree with the usual definitions of cohomology of G when A is abelian. These constructions are also functorial in A and G . If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence of non-abelian G -modules, then we have the following exact sequence of pointed sets

$$H^0(G, A) \rightarrow H^0(G, B) \rightarrow H^0(G, C) \rightarrow H^1(G, A) \rightarrow H^1(G, B) \rightarrow H^1(G, C)$$

when the group G is Galois group of a (not necessarily finite) field extension and A a topological G -module such that

$$A = \bigcup A^H$$

when H runs through the open normal subgroups of G , we define

$$H^1(G, A) := \varinjlim H^1(G/H, A^H)$$

There is a simple geometric interpretation for $H^1(G, A_K)$: It is the set of classes of principal homogeneous spaces for A , defined over k which have a rational point over K , [6] page 123. As an example, take $G = \mathbb{Z}/2 = \text{Gal}(\mathbb{C}/\mathbb{R})$ acting naturally on a set of tensors $A_{\mathbb{R}}$. Then any element in $H^1(\mathbb{Z}/2, A_{\mathbb{C}})$ is determined by an involution map $-1 : a_{-1} \mapsto * \in A_{\mathbb{R}}$. This shows $H^1(\mathbb{Z}/2, A_{\mathbb{C}}) = A_{\mathbb{R}}$, [6].

If V be a vector space over k , provided with a fixed tensor of x of type (p, q) , i.e. $x \in \bigotimes^p V \otimes \bigotimes^q V^*$ where V^* is the dual of V . two pairs $(V, x), (V, x')$ are called k -isomorphic if there is a k -linear isomorphism $f : V \rightarrow V'$ such that $f(x) = x'$. Denote by A_k the group of these automorphisms. Let K/k be a Galois extension with Galois group G . Write $E_{V,x}(K, k)$ for the set of k -isomorphism classes that are K -isomorphic to (V, x) . The group G acts on V_K by $s.(x \otimes \lambda) = x \otimes s.\lambda$. It also acts on $f : V \rightarrow V'$ by $s.f = s \circ f \circ s^{-1}$. If we put

$$p_s = f^{-1} \circ s \circ f \circ s^{-1}, \quad s \in G$$

the map $s \mapsto p_s(f)$ is a 1-cocycle in $H^1(G, A_K)$, [6].

Theorem 3.1. ([6] page 153) *The map $\theta : E_{V,x}(K/k) \rightarrow H^1(G, A_K)$ defined by*

$$f \mapsto p_s(f)$$

is a bijection.

Theorem 3.2. ([6] page 153) *The set $H^1(G, \text{Aut}(Q, K))$ is in bijective correspondence with the classes of quadratic k -forms that are K -isomorphic to Q .*

The above definition can be generalized in this way that instead of considering a single tensor $T^{p,q} = \bigotimes^p V \otimes \bigotimes^q V^*$ one may consider a sum of such tensors that is a subset T as

$$T \subset T^{\bullet,\bullet} = \bigoplus_{p,q} T^{p,q}$$

The proofs will proceed exactly the same and we obtain

Theorem 3.3. *Assume T is a set of tensors for a vector space V . The map $\theta : E_{V,T}(K/k) \rightarrow H^1(G, A_{T,K})$ defined by*

$$f_T \mapsto p_s(f_T)$$

is a bijection, where $A_{T,K}$ is the group of K -automorphisms of all the tensors in T .

(2) To begin with let V be finite dimensional \mathbb{Q} -vector space, and Q a non-degenerate bilinear map $Q : V \otimes V \rightarrow \mathbb{Q}$ which is $(-1)^n$ -symmetric for some fixed n . A Hodge structure is given by a representation $\phi : \mathbb{U}(\mathbb{R}) \rightarrow \text{Aut}(V, Q)_{\mathbb{R}}$. The Mumford-Tate group of the Hodge structure ϕ denoted $M_{\phi}(R)$ is the smallest \mathbb{Q} -algebraic subgroup of $G = \text{Aut}(V, Q)$ with the property $\phi(\mathbb{U}(R)) \subset M_{\phi}(R)$. M_{ϕ} is a simple, connected, reductive \mathbb{Q} -algebraic group. If $F^{\bullet} \in \check{D}$ the Mumford-Tate group $M_{F^{\bullet}}$ is the subgroup of $G_{\mathbb{R}}$ that fixes the Hodge tensors.

We are going to investigate the relation between nonabelian cohomology discussed above, and classifying spaces for Hodge structures. By definition $H^0(G_{\mathbb{R}}, Hg_{F^{\bullet}}^{\bullet,\bullet}) = M_{F^{\bullet}}$. The relation $H^1(\mathbb{Z}/2, G_{\mathbb{C}}) = G_{\mathbb{R}}$ is trivial. By applying Theorem 4.3 to the above construction we get the following results.

Theorem 3.4. (1) $H^1(\mathbb{Z}/2, M_{F^{\bullet}}) = M_{\phi}$.

(2) $H^1(\text{Gal}(\mathbb{C}/\mathbb{Q}), \text{Aut}(Q, \mathbb{C})) = \check{D}$

(3) $H^1(\text{Gal}(\mathbb{R}/\mathbb{Q}), \text{Aut}(Q, \mathbb{R})) = D$

Proof. (1) follows from the example at (1), and the fact that M_{ϕ} is the subgroup of G that fixes pointwise the algebra of Hodge tensors, cf. [MGK].

(2) Follows from Theorem 3.3, noting that $\check{D} = G_{\mathbb{C}}/\text{Stab}_{G_{\mathbb{C}}}(F_0)$.

(3) This also follows from Theorem 3.3, and similar identity $D = G_{\mathbb{R}}/Stab_{G_{\mathbb{R}}}(F_0)$. \square

We have the short exact sequence

$$0 \rightarrow \mathbb{Z}/2 \rightarrow Gal(\mathbb{C}/\mathbb{Q}) \rightarrow Gal(\mathbb{R}/\mathbb{Q}) \rightarrow 0$$

where the second non-zero map is the restriction

$$\begin{aligned} 0 \rightarrow H^1(\mathbb{Z}/2, Aut(Q, \mathbb{C})^{Gal(\mathbb{R}/\mathbb{Q})}) &\rightarrow H^1(Gal(\mathbb{C}/\mathbb{Q}), Aut(Q, \mathbb{C})) \\ &\rightarrow H^1(Gal(\mathbb{R}/\mathbb{Q}), Aut(Q, \mathbb{C}))^{\mathbb{Z}/2} \end{aligned}$$

where the second map is the restriction. The first map is called inflation map. The first item is $Aut(Q, \mathbb{R})$, and the second item is \check{D} by Theorem 3.2. Thus we have

$$0 \rightarrow Aut(Q, \mathbb{R}) \rightarrow \check{D} \rightarrow H^1(Gal(\mathbb{R}/\mathbb{Q}), Aut(Q, \mathbb{C}))^{\mathbb{Z}/2}$$

as exact sequence of sets with distinguished unit elements.

Theorem 3.5. (1) $H^1(\mathbb{U}_{\mathbb{R}}, G_{\mathbb{R}}) = D$, where $G_{\mathbb{R}}$ acts on $G_{\mathbb{R}} = Aut(Q, \mathbb{R})$ by $g : T \mapsto g^t T g$.

(2) $H^1(\mathbb{U}(\mathbb{R}), Aut(\mathfrak{g}_{\mathbb{R}}, B)) = D$, where $\mathbb{U}_{\mathbb{R}}$ acts by $g.X = g^t X g$, $X \in \mathfrak{g}_{\mathbb{R}}$, and B is the killing form.

Proof. (1) The cocycle condition is equivalent to $\mathbb{U}_{\mathbb{R}} \rightarrow G_{\mathbb{R}}$ being a homomorphism and the boundary condition is when two such homomorphism are conjugate by an element of $G_{\mathbb{R}}$. Then, the theorem is consequence of that, D is isomorphic to the set of conjugacy classes of the isotropy group of a fixed Hodge structure.

(2) Again the cocycle condition is the cocycles are $Hom(\mathbb{U}_{\mathbb{R}}, Aut(B))$, and the coboundary condition becomes when two such homomorphisms are conjugate by an automorphism of B . Regarding B as a tensor then the theorem follows from the known fact that M_{ϕ} is the subgroup of G with the property that M_{ϕ} -stable subspaces $W \subset T_{\phi}^{a,b}$ are exactly the sub-Hodge structures of these tensor space. \square

For each point $\phi \in D$, the adjoint representation

$$Ad\phi : \mathbb{U}(\mathbb{R}) \rightarrow Aut(\mathfrak{g}_{\mathbb{R}}, B)$$

induces a Hodge structure of weight 0 on \mathfrak{g} . This Hodge structure is polarized by the killing form $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$. and it is a sub-Hodge structure of $\check{V} \times V$. Therefore (2) is analogue of (1) applied to this vector space.

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